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# Benatti–Narnhofer–Sewell Quantum Arnol'd Cat Map: Physical Interpretation

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A Tomita (standard) Hilbert space representation of a physical system with a toral phase space is presented, both for the classical and for the quantal case. The dynamics of such a system, subject to a regular series of linear kicks, is derived in a discrete-time form, and is shown to be identical to the Benatti–Narnhofer–Sewell quantum Arnol'd cat map in the quantum case. It is shown that this quantum system is chaotic in the sense of having exponentially decaying autocorrelation functions.

## 1. INTRODUCTION

The Arnol'd cat map is one of the simplest possible prototypes of a chaotic Hamiltonian system. It can be interpreted as a discrete-time representation of a mechanical system subject to regular kicks by a repulsive linear force (Ford *et al.*, 1991). The mathematical simplicity of the system combined with its strong ergodic and stochastic properties makes it a suitable object for the study of the behavior of a quantum system with a chaotic classical counterpart. However, the task of finding the proper method of quantization for this system is far from trivial, due to the confinement of both the position and the momentum space. Thus, there exist several quantum cat map models (e.g., Berry *et al.*, 1979; Ford *et al.*, 1991). One of these is presented by Benatti *et al.* (1991), hereafter referred to as the BNS model. This model is further described by Benatti (1993). An interesting aspect of this model is that it retains chaotic properties after quantization, in particular for certain values of $\hbar$. BNS do not comment on the possible connection between their abstract model and the physical linearly kicked system, as they restrict themselves to considerations of the mathematical properties of the model.

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In this paper, I investigate the relation between the abstract BNS model and the physical linearly kicked system, demonstrating that the model can be considered as a quantized version of such a system by identifying the operators representing the physical observables. Then I present a further investigation of the stochastic properties of the quantum system. While BNS prove that the system is only a K-system in the strict, algebraic sense for certain values of $\hbar$, I show that for *all* $\hbar$ the system is chaotic in the weaker sense of having an exponential decay of the autocorrelation. I conclude that the quantum correspondent to a confined classical system may retain strong stochastic properties, although the strict confinement turns out to be lost due to the uncertainty principle.

In the following section, I summarize the structure of the classical and quantal toral phase space system, considering in particular the operators having a physical interpretation. In Section 3, I derive the Arnol'd cat map dynamics from physical assumptions. Section 4 shows that the autocorrelation function decays exponentially.

## 2. THE TOMITA REPRESENTATION OF THE TORAL SYSTEM

First, we present a model of a mechanical system with the torus $T^2 = [0, 1)^2$, equipped with addition and multiplication modulo 1, as a phase space. The area of the torus has the dimension of action. To define an appropriate Tomita, or standard, representation of this system, we apply essentially the definition of the Wigner representation of Grelland (1993), with a few modifications to deal with the confined phase space.

The Tomita Hilbert space $\mathsf{H}$ of the system is characterized by the continuous Wigner basis

$$\mathsf{B}_W = \{|qp] | q, p \in [0, 1]\} \tag{1}$$

where $|...]$ is the appropriate notation for a ket vector of this kind of representation. This basis gives rise to a representation in terms of the state functions $f(q, p) = [qp|f]$. We also introduce the countable Fourier basis

$$\mathsf{B}_F = \{|kl] | k, l \in \mathbf{Z}; [qp|kl] = e^{2\pi i(qk+pl)}\} \tag{2}$$

On $\mathsf{H}$ we define the classical position and momentum $Q_o$, $P_o$ by the operators

$$Q_o|qp] = q|qp] \tag{3}$$

$$P_o|qp] = p|qp] \tag{4}$$

Note that for this system these operators are bounded and are defined on the whole of $\mathsf{H}$. The (mixed) states of the classical system are represented by positive, real, normalized functions $[qp|f]$ corresponding to probability densities $P(q, p) = [qp|f]^2$ in phase space.

To obtain the quantum position and momentum, it is convenient to define the auxiliary operators

$$D|kl] = -2\pi k|kl] \tag{5}$$

$$E|kl] = -2\pi l|kl] \tag{6}$$

These operators are unbounded and have a restricted domain. In the Wigner basis they act like derivation operators

$$[qp|D|f] = i\, d/dq\, [qp|f] \tag{7}$$

$$[qp|E|f] = i\, d/dp\, [qp|f] \tag{8}$$

on the domains

$$\mathsf{D}(D) = \{|f]|[qp|f] \in AC[0, 1] \otimes L^2[0, 1]\ \&\ [0p|f] = [1p|f]\} \tag{9}$$

$$\mathsf{D}(E) = \{|f]|[qp|f] \in L^2[1, 0] \otimes AC[0, 1]\ \&\ [q0|f] = [q1|f]\} \tag{10}$$

where $AC[1, 0]$ is the space of absolutely continuous functions on the interval. Both these operators have the countable, unbounded, nondegenerate spectrum $\sigma = 2\pi\mathbf{Z}$.

We are now in a posiition to define the quantum observables by the operators

$$Q = Q_o + (\hbar/2)E \tag{11}$$

$$P = P_o - (\hbar/2)D \tag{12}$$

with the Born–Heisenberg commutation property

$$[Q, P]|f] = i\hbar|f] \qquad \forall|f] \in \mathsf{D}([Q, P]) \tag{13}$$

Observe that $\hbar$ is measured in units of the phase space area. For these operators, the domains are

$$\mathsf{D}(Q) = \mathsf{D}(E) \tag{14}$$

$$\mathsf{D}(P) = \mathsf{D}(D) \tag{15}$$

$$\begin{aligned}\mathsf{D}([Q, P]) &= \{|f]|P|f] \in \mathsf{D}(Q)\ \&\ Q|f] \in \mathsf{D}(P)\} \\ &= \{|f]|[qp|f] \in AC[0, 1] \otimes AC[0, 1]; \\ &\quad [00|f] = [01|f] = [10|f] = [11|f] = 0\}\end{aligned} \tag{16}$$

The spectra of the position and momentum operators are determined by the size of the phase space, reflected in the choice of units such that the phase space becomes the unit square. If the size is such that $\hbar/2 \le 1$, the spectra of $Q$ and $P$ are both $\mathbf{R}$. This lack of physical confinement reflected in the spectra of $Q$ and $P$ in the quantum model may be surprising, but it is the only possibility consistent with the uncertainty relations. *One cannot physically strictly confine a quantum system, since it can penetrate any potential barrier, but one can construct quantum systems with a confined classical analogue.*

The connection between the representation described and the operator-based formalism of BNS is obtained by defining the unitary group $\{W(n, m)|n, m \in \mathbf{Z}\}$:

$$W(n, m) = \exp[2\pi(Qn + Pm)] \tag{17}$$

which generates the same group as does the pair $\{Q, P\}$. It is easily seen that these operators have the ladder property when applied to the Fourier basis,

$$W(n, m)|k, l] = |k + n, l + m] \tag{18}$$

The group $\{W_o(n, m)\}$ of BNS contains the operators

$$W_o(n, m) = \exp[2\pi(Q_o n + P_o m)] \tag{19}$$

which generates the commutative "classical algebra." As $\hbar \to 0$, $W(n, m) \to W_o(n, m)$ strongly.

## 3. THE EQUATION OF MOTION

The linearly kicked system (LKS) (also called "the kicked oscillator") has the time-dependent Hamiltonian (Ford *et al.*, 1991; Benatti, 1993)

$$H(t) = P^2/2m - kQ^2/2 \, \Sigma_j \, \delta(j - t/T) \tag{20}$$

The force switched on by the delta functions at regular intervals is linear and repulsive, generated by a negative quadratic potential. Applying the Tomita representation of both classical and quantum mechanics, an operator formalism is used in both cases. Thus, the following operator-based derivation of the equations of motion applies to both cases, as no commutator is involved. For the classical case, our derivation is equivalent to that of Ford *et al.* (1991). For the quantum case, the derivation leads to the same result as the one presented by Berry *et al.* (1979) in the Heisenberg picture, where the representation is unspecified. We show, however, that it is unnecessary to assume *a priori* any "turning off" of the kinetic term during the kick, as the contribution from this term proves to be negligible.

Applying the operator formalism, we have the operator generating a one-step time evolution:

$$U = \exp[-(i/\hbar) \int_{0-\Delta t}^{T-\Delta t} H(t) - H'(t) \, dt] \tag{21}$$

where we will end up by neglecting the small quantity $\Delta t$. The time evolution operator contains the term $H' = H(Q', P')$, where

$$Q' = Q_o - (\hbar/2)E \tag{22}$$

$$P' = P_o + (\hbar/2)D \tag{23}$$

analogous to the Wigner representation of Grelland (1991). Thus,

$$H(t) - H'(t) = -(h/m)P_oD - hkQ_oE\,\delta(t/T) \tag{24}$$

The time scale is chosen such that the period starts at $t - \Delta t$, i.e., immediately before the kick. Thus,

$$|n + 1] = U|n] \tag{25}$$

where $|n]$ is the state at the end of period $n$. We divide the period into two time subintervals:

$$U(0 - \Delta t, T - \Delta t) = U(\Delta t, T - \Delta t)U(-\Delta t, \Delta t) \equiv U_2U_1 \tag{26}$$

where

$$\begin{aligned} U_1 = U(-\Delta t, \Delta t) &= \exp[-(i/\hbar) \textstyle\int_{-\Delta t}^{\Delta t} H(t) - H'(t)\, dt] \\ &= \exp[-(i/h) \textstyle\int_{-\Delta t}^{\Delta t} \{(h/m)P_oD - hkQ_oE\,\delta(t/T)\}\, dt] \\ &= \exp[-\{m^{-1}P_oD\, 2\Delta t - kTQ_oE\}] \end{aligned} \tag{27}$$

since $\delta(t/T) = T\delta(t)$. We choose effectively $\Delta t = 0$; thus the first term of the exponent can be neglected, and we obtain

$$U_1 = \exp[ikTQ_oE] \tag{28}$$

The second term is

$$\begin{aligned} U_2 = U(\Delta t, T - \Delta t) &= \exp[-(i/\hbar) \textstyle\int_{\Delta t}^{T-\Delta t} H(t) - H'(t)\, dt] \\ &= \exp[(i/m)TP_OD] \end{aligned} \tag{29}$$

neglecting terms proportional to $\Delta t$. As expected, when the potential function does not contain higher powers than two, the resulting "Liouville" operators have the same form as the classical Liouvilleans. However, they act on a different state space (cone of state functions). In the representation generated by the $\mathsf{B}_\mathsf{W}$ basis, we obtain the operator forms

$$U_1 = \exp[-(Tp/m)\, d/dq] \tag{30}$$

$$U_2 = \exp[-kTq\, d/dp] \tag{31}$$

Hence (assuming addition modulo 1 of the coordinates)

$$[qp|n+1] = [qp|U|n] = [((1+kT^2/m)q - (T/m)p, -kTq+p)|n]$$
$$= [\theta^{-1}(q, p)|n] \tag{32}$$

where $\theta(q, p) = (q + (T/m)p, kTq + (1 + (kT^2/m)p)$ is the Arnol'd cat map when $T/m = kT = 1$. We can avoid using the inverse map $\theta^{-1}$ by considering the inverse dynamics, generated by $U^{-1}$. In that case we obtain the dynamical map of BNS.

## 4. EXPONENTIAL DECAY OF THE CORRELATION FUNCTION

The classical LKS is chaotic (Arnol'd and Avez, 1968). It is a Bernoulli system, hence a Kolmogorov system (K-system), and hence a mixing ergodic system. It is also chaotic in the sense of having an exponentially decaying correlation function. The correlation function is the sequence of matrix elements of the operator sequence

$$R_n = U^n - |\mathbf{0}][\mathbf{0}| \tag{33}$$

where $|\mathbf{0}] = |k, l]$, with $k = l = 0$, is the state corresponding to uniform distribution on the phase space, $[qp|\mathbf{0}] = 1$. The exponential decay property,

$$[f|R_n|g] \to Ae^{-an} \tag{34}$$

for large $n$ is characteristic of a white noise process, or a deterministic process with local exponential separation of the phase space paths, combined with confinement. Thus this property is a possible definition of chaos in this formal framework. However, it is weaker than the K-property.

Benatti *et al.* (1991) have shown that quantum LKS has the following properties:

(i) It has the decay property, $[f|R_n|g] \to 0$.

(ii) It is mixing.

(iii) It is an algebraic K-system for rational values of $h$ (Planck's constant).

(iv) It is not an algebraic, nor an entropic, K-system for irrational values of $h$.

We will strengthen (i) by showing that

$$[f|R_n|g] \to Ae^{-an} \tag{35}$$

for the quantum (general) case.

We introduce the vector notation $\boldsymbol{k} = (k, l)$, $\boldsymbol{z} = (q, p)$; $\boldsymbol{A} = (1, 1; 1, 2)$ is the matrix of the Arnol'd cat map. $\boldsymbol{A}$ has the eigenvalues $\lambda_\pm =$

$(3 \pm 5^{1/2})/2$. Note that $\lambda_- < 0$, such that $\lambda_-^n \to 0$ as $n \to \infty$. We define the orthogonal matrix

$$U = (t_+, t_-; t_+(1 + 5^{1/2})/2, t_-(1 - 5^{1/2})/2) \tag{36}$$

where

$$t_\pm = \pm ((5 \pm 5^{1/2})/2)^{-1/2} \tag{37}$$

such that $U^T A U$ is diagonal. We define the correlation functions as the matrix elements

$$\begin{aligned} [\boldsymbol{k}'|U^n|\boldsymbol{k}] &= \int_{[0,1]} [\boldsymbol{k}'|z]\, dz\, [z|U^n|\boldsymbol{k}] \\ &= \int_{[0,1]} \exp[2\pi i(\boldsymbol{k} \cdot \boldsymbol{A}^n z - \boldsymbol{k}' \cdot z)]\, dz \\ &\propto (2\pi a_+{}^n)^{-2} = 4\pi^2 e^{-an} \end{aligned} \tag{38}$$

where $a = 2 \ln \lambda_+$. In the derivation we have assumed that for sufficiently large $n$, $U^T A U$ can be replaced by $(\lambda_+, 0; 0, 0)$. Since $[\boldsymbol{k}'|\boldsymbol{0}][\boldsymbol{0}|\boldsymbol{k}] = 0$ except for $\boldsymbol{k}' = \boldsymbol{k} = \boldsymbol{0}$, in which case $[\boldsymbol{k}'|U^n|\boldsymbol{k}] = [\boldsymbol{k}'|\boldsymbol{0}][\boldsymbol{0}|\boldsymbol{k}] = 1$, we conclude that (38) is valid for all $\boldsymbol{k}'$, $\boldsymbol{k}$.

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